Fokker Planck equation, Poincare inequaltiy, and convergence of Markov Process

Jian Li

IIIS Tsinghua

ATCS 2023

Recall Fokker Planck equation

- Consider a diffusion process on $\mathbb{R}^d$ with time-independent drift and diffusion coefficients. The Fokker-Planck equation is

$$\frac{\partial p}{\partial t} = -\sum_{j=1}^{d} \frac{\partial}{\partial x_j}(a_i(x)p) + \frac{1}{2}\sum_{i,j=1}^{d} \frac{\partial^2}{\partial x_i \partial x_j}(b_{ij}(x)p),\ t > 0,\ x \in \mathbb{R}^d,$$

$$p(x,0) = f(x), \quad x \in \mathbb{R}^d.$$

Fokker Planck equation for O-U proces: $dX_t = -\alpha X_t dt + \sqrt{2D} dW_t$

Set $a(t,x) = -\alpha x$, $b(t,x) \equiv 2D > 0$:

$$\frac{\partial p}{\partial t} = \alpha \frac{\partial (xp)}{\partial x} + D\frac{\partial^2 p}{\partial x^2}.$$

Fokker Planck equation

$$dX_t = -\nabla V(X_t)\,dt + \sqrt{2D}\,dW_t.$$

- The corresponding FP equation is:

$$\frac{\partial p}{\partial t} = \nabla \cdot (\nabla V p) + D\Delta p.$$

The stationary distribution of the above Markov process is the following Gibbs distribution:

$$p(x) = \frac{1}{Z} e^{-V(x)/D}$$

one can verify it satifies FP equation

where the normalization factor Z *is the* **partition function**

$$Z = \int_{\mathbb{R}^d} e^{-V(x)/D}\,dx.$$

A normalized version

- It is more convenient to normalize the solution of the
- Fokker-Planck equation wrt the invariant distribution

let $\rho(x)$ *be the Gibbs distribution*

Define $h(x,t)$ *through*

$$p(x,t) = h(x,t)\rho(x).$$

Then the function h *satisfies the* **backward Kolmogorov equation***:*

$$\frac{\partial h}{\partial t} = -\nabla V \cdot \nabla h + D\Delta h, \quad h(x,0) = p(x,0)\rho^{-1}(x).$$

Proof. The initial condition follows from the definition of h. We calculate the gradient and Laplacian of p:

$$\nabla p = \rho \nabla h - \rho h D^{-1} \nabla V$$

and

$$\Delta p = \rho \Delta h - 2\rho D^{-1} \nabla V \cdot \nabla h + h D^{-1} \Delta V \rho + h |\nabla V|^2 D^{-2} \rho.$$

We substitute these formulas into the FP equation to obtain

$$\rho \frac{\partial h}{\partial t} = \rho \Big(- \nabla V \cdot \nabla h + D \Delta h \Big),$$

from which the claim follows. □

The self-adjoint generator

- Consider the Hilbert space with the following inner product

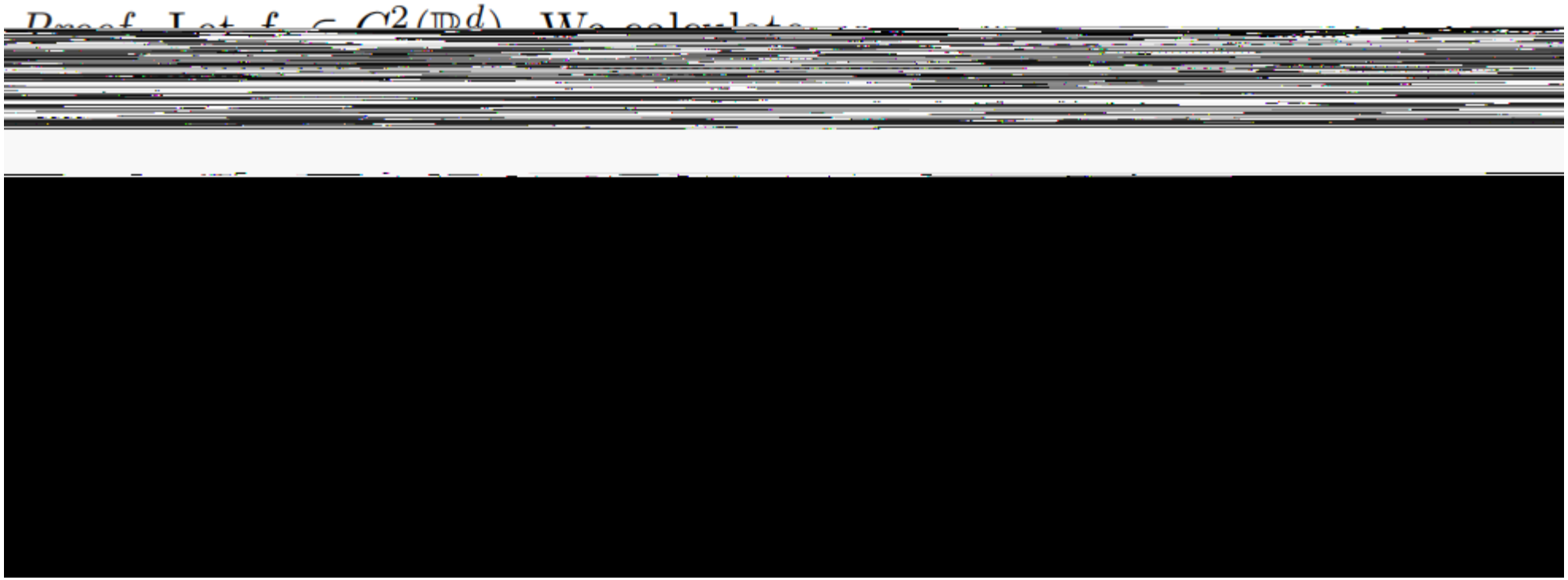
$$(f,h)_\rho = \int_{\mathbb{R}^d} fh\rho(x)\,dx.$$

Proposition 3. *Assume that $V(x)$ is a smooth potential and assume that condition (7) holds. Then the operator*

$$\mathcal{L} = -\nabla V(x)\cdot\nabla + D\Delta$$

is self-adjoint in L^2_ρ. Furthermore, it is non-positive, its kernel consists of constants.

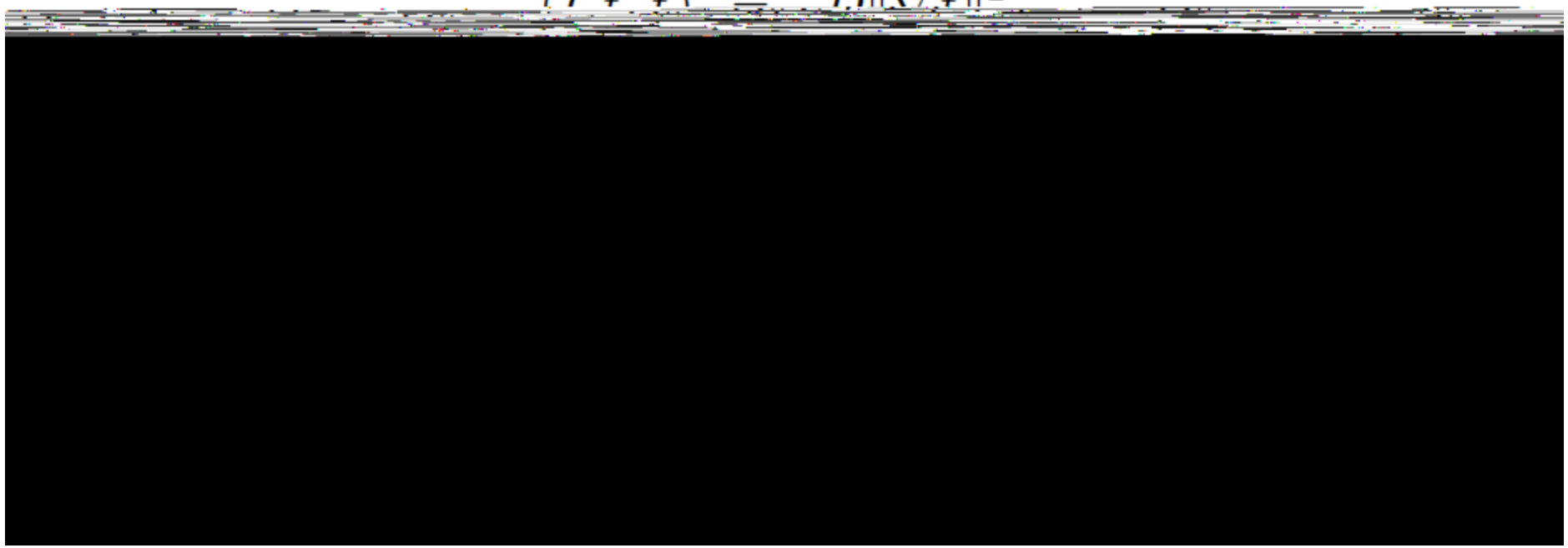
The self-adjoint generator



The self-adjoint generator

If we set $f = h$ in the above equation we get

$$(\mathcal{L}f, f) = -D\|\nabla f\|^2$$



Proof.

As analysed above, if P is symmetric, it has orthonormal eigenvectors $v_1, \dots, v_N$ such that any distribution q over Ω can be expressed as

$$q = \sum_{i=1}^{N} c_i v_i = \pi + \sum_{i=2}^{N} c_i v_i$$

with $c_i = q^T v_i$, and

$$qP^t = \pi + \sum_{i=2}^{N} c_i \lambda_i^t v_i.$$

When q is a distribution, i.e., q is a nonnegative vector and $\|q\|_1 = 1$, it holds that $c_1 = q^T v_1 = \frac{1}{\sqrt{N}}$ and $c_1 v_1 = \left(\frac{1}{N}, \dots, \frac{1}{N}\right) = \pi$, thus

[redacted]

Thus,

$$\begin{aligned}
\|qP^t - \pi\|_1 &= \left\| \sum_{i=2}^{N} c_i \lambda_i^t v_i \right\|_1 \\
&\le \sqrt{N} \left\| \sum_{i=2}^{N} c_i \lambda_i^t v_i \right\|_2 && \text{(Cauchy-Schwarz)} \\
&= \sqrt{N} \sqrt{\sum_{i=2}^{N} c_i^2 \lambda_i^{2t}} \\
&\le \sqrt{N} \lambda_{\max}^t \sqrt{\sum_{i=2}^{N} c_i^2} \\
&= \sqrt{N} \lambda_{\max}^t \|q\|_2 \\
&\le \sqrt{N} \lambda_{\max}^t.
\end{aligned}$$

The last inequality is due to a universal relation $\|q\|_2 \le \|q\|_1$ and the fact that q is a distribution.